

# Holographic Entanglement Entropy for Excited States in Two Dimensional CFT

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## Abstract

We use holographic methods to study the entanglement entropy for excited states in a two dimensional conformal field theory. The entangling area is a single interval and the excitations are produced by in and out vertex operators with given scaling dimensions. On the gravity side we provide the excitations by turning on a scalar field with an appropriate mass. The calculation amounts to using the gravitational background, with a singular boundary, to find the one point function of the vertex operators. The singular boundary is taken care of by introducing a nontrivial UV regulator surface to calculate gravitational partition functions. By means of holographic methods we reproduce the field theory results for primary excitations.

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# Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Entanglement Entropy in Quantum Field Theory</b>	<b>3</b>
2.1	Review of EE in QFT . . . . .	3
2.2	Entanglement entropy for primary excitations in 2-d CFT . . . . .	4
<b>3</b>	<b>Holographic Entanglement Entropy</b>	<b>5</b>
3.1	<i>Ryu-Takayanagi's</i> proposal . . . . .	5
3.2	Direct holographic calculation . . . . .	5
3.3	The role of the UV regulator . . . . .	6
<b>4</b>	<b>Holographic calculation of the entanglement entropy for excited states</b>	<b>7</b>
4.1	Holographic Setup . . . . .	7
4.2	Calculations . . . . .	8
<b>5</b>	<b>Discussion</b>	<b>14</b>

## 1 Introduction

There are many reasons why Entanglement Entropy (EE) has attracted a lot of attention in the last few years. On the one hand it captures information about the quantum structure of physical systems, to the extent that it can be used as an order parameter to identify quantum phases at zero temperature. On the other hand there are several hints that this quantity can address outstanding questions in quantum gravity.

Incidentally, EE first attracted considerable interest, [1–3], when it was shown to have similarities with the entropy of black holes through the area law [4, 5]. This brought about the possibility that black hole entropy might be explained as arising from entanglement between degrees of freedom on the two sides of horizon.

Although these early hopes and speculations did not survive further tests and expectations, in recent years new venues and frameworks have been created to address and formulate old questions. Holography, and gauge/gravity correspondence in a broader sense, has been the subject of intense research activity in this regard [6–8]. It is specially in this context that EE has proved to be very useful.

The proposal of [9] (see also [10–12] for reviews and references), gives a holographic prescription for calculation of EE. This has enabled one to predict how this quantity behaves in the strong coupling regime of certain quantum field theories, a domain which is not accessible by the usual perturbative methods. The results obtained this way suggest that the area law, governing EE, remains generally valid in strongly coupled systems. Furthermore, violations of the area law and/or the form of the subleading contributions to EE contain various information about the strongly coupled system, information such as whether the system has a Fermi surface, whether it is in a (de)confined phase, whether it is a (non)local theory, etc. (see [13–38] for some related works, reviews and references).

On the other hand, the above mentioned proposal gives a geometric interpretation for EE. This may lead one into insights about the mechanism of holography, how gravity comes into play, how

additional directions of space are generated in a gravitational description, what regions of geometry are describing which experiments in field theory and so on and so forth.

The idea of entanglement arises when one considers two or more disjoint sets of degrees of freedom in a system. These degrees of freedom can be far apart spatially and hence may be considered as living in different regions of space. If there is a nonzero entanglement between two such regions, local measurements and observations in one region are affected by those in the other instantaneously. The effect is obviously not carried by any messenger, rather, it is the result of the quantum structure of the state, that is, the way the overall quantum state of the system is described in terms of the local quantum states in each region.

An observer that is confined to one of the regions will measure effects which he cannot explain by the degrees of freedom accessible to him and hence he will have some sort of *ignorance* about the accessible region. This ignorance or lack of information is quantified by the EE.

Lack of information can also be caused by statistical distribution of states in a system such as the case for thermal ensembles. In such situations EE will no longer be a useful measure of quantum entanglement and thus one usually studies this quantity when the system is in a pure state. Most of the research on this subject has focused on the case where this pure state is the ground state of the theory. In this article we are interested in excited pure states.

As a first attempt one can consider primary excitations in a two dimensional CFT and study the EE of a single interval. This problem has been addressed in [39], [40] and [41] where three different methods have been used for calculations. The first paper follows the approach of Holzhey, Larsen and Wilczek [13], the second uses methods developed by Calabrese and Cardy [14] and the third one applies techniques in symmetric orbifolding [42]. In this work we will address this problem by holography.

On the field theory side, one resorts to the replica trick for calculation of the Renyi entropy which gives EE as a limit. This is achieved by calculating the partition function of the theory on a certain Riemann surface. The excitations are introduced by inserting appropriate in and out vertex operators on this surface. One then uses suitable conformal transformations to move over to a smooth manifold on which the correlation function of the vertex operators is calculated. The complexities of the Riemann surface, which contain the nontrivial information of entanglement, are encoded in the transformation to the smooth surface.

In this work we make a parallel line of calculations on the gravitational side. We construct the space time out of the boundary data and turn on certain fields with suitable boundary conditions to account for excitations. This space will have a complicated Riemann surface as the boundary. We will replace it with a smooth surface and encode its complexities in a nontrivial UV regulator surface near the boundary. The regulator will then be used to calculate the gravitational partition function.

Our holographic calculations rely on the well developed techniques of holographic renormalisation. We will use approximations to write down the renormalized action on our nontrivial regulator surface. The approximations we will use will turn out to be equivalent to the assumption that on the field theory side some specific operators have been turned on. The conformal properties of these operators are those for primary excitations. From another point of view we may conclude that once we “demand” our approximations to be exact, we have effectively given a bulk “definition” for the field theory primary operators.

By the holographic calculation we therefore reproduce the field theory results which have been obtained for *primary* excitations of CFT.

In the following chapters we will first review EE in field theory and its holographic description. In particular we will mention the role of a regulator surface in the holographic calculation. We will also briefly outline the field theory calculation of EE in presence of primary excitations. We will

then use gravity to address this problem. We close with conclusions and discussions.

## 2 Entanglement Entropy in Quantum Field Theory

In this section we first give a short review of the basics of EE in field theory and will then briefly outline the field theoretic calculation of EE of a single interval in a two dimensional conformal field theory in presence of primary excitations. The latter will be addressed in a holographic setup in subsequent sections.

### 2.1 Review of EE in QFT

Consider a quantum mechanical system in a pure state  $|\psi\rangle$  and correspondingly with the density operator  $\hat{\rho} = |\psi\rangle\langle\psi|$ . Divide the system into two subsystems  $A$  and  $B$  and define the *reduced* density operator of  $A$  as  $\hat{\rho}_A = \text{tr}_B \hat{\rho}$ . Alternatively one could take the trace over the  $A$  degrees of freedom and define  $\hat{\rho}_B$ . Generically the reduced density operator will no longer be pure and one can attribute entropy to it. The EE of subsystem  $A$  is defined as the *Von-Neumann* entropy of this operator

$$S_A = -\text{tr}_A \hat{\rho}_A \log \hat{\rho}_A. \quad (1)$$

It is also possible to define and calculate EE in the context of quantum field theory. A usual approach is to define a useful mathematical quantity, called *Renyi Entropy*, by the *Replica Trick* as

$$S_n[A] \equiv \frac{1}{1-n} \log \text{tr}_A \hat{\rho}_A^n. \quad (2)$$

It is easy to see that EE can be obtained as  $S_A = \lim_{n \rightarrow 1} S_n[A]$  whenever the limit exists. This quantity can be represented in terms of a path integral by considering  $n$  copies of the world volume of the original theory,  $\mathcal{M}$ , and glueing them along the entangling subspaces in a cyclic order. This results in a space, which we denote by  $\mathcal{R}_n$ , and which has singularities on the boundaries of entangling subspaces. The path integral on  $\mathcal{R}_n$  is denoted by  $Z_{\mathcal{R}_n}$  (or  $Z_n$  in short) and is defined as

$$Z_{\mathcal{R}_n} = \int [d\varphi(x)] e^{-S[\varphi]}, \quad x \in \mathcal{R}_n, \quad (3)$$

where  $n$  is the replica number and  $\varphi$  denotes dynamical fields collectively. One can then calculate the Renyi entropy as

$$S_n = \frac{1}{1-n} \log \frac{Z_n}{Z_1}, \quad (4)$$

where  $Z_1$  represents the partition function of the original unreplicated theory.

Due to the complicated singular topology of the Riemann surface it is usually very difficult to calculate this partition function directly. One can go around this by transferring the geometric complexities of the world volume into the geometry of target space. That is, one considers the original nonsingular world volume but instead introduces  $n$  copies of the target space fields,  $\varphi_i$  ( $i = 1, 2, \dots, n$ ), on that. Instead of gluing the world volumes, one now restricts the fields to satisfy certain conditions along the entangling subspaces

$$Z_{res} = \int_{res} [d^n \varphi(x)] e^{-S[\varphi_1, \dots, \varphi_n]}, \quad x \in \mathcal{M}, \quad (5)$$

where the subscript *res* stands for restrictions on fields. Note that these restrictions replace the non-trivial geometry of  $\mathcal{R}_n$ . One way to impose the restrictions is to insert the so called *twist operators*

at the boundaries of entangling subspaces and calculate an unrestricted integral

$$Z_{Twist} = \int_{unres} [d^n \varphi(x)] e^{-S[\varphi_1, \dots, \varphi_n]} \prod \sigma_k \dots, \quad x \in \mathcal{M}, \quad (6)$$

where  $\sigma_k$  are the twist operators that open and close the branch cuts along the entangling subspaces and enforce the restrictions through their Operator Product Expansion (OPE) with fields. An alternative way of imposing the restrictions is to move over to the covering space of the fields, denoted by  $\mathcal{M}_C$ , with a suitable coordinate transformation and perform the calculations on this manifold

$$Z_{\mathcal{M}_C} = \int [d\varphi(x)] e^{-S[\varphi]}, \quad x \in \mathcal{M}_C. \quad (7)$$

On the covering space the restrictions on fields in the integration are taken care of by the geometry of  $\mathcal{M}_C$ . This space is now a smooth manifold and the complexities of the Riemann surface are encoded in the transformation to  $\mathcal{M}_C$ <sup>3</sup>.

## 2.2 Entanglement entropy for primary excitations in 2-d CFT

In this section we briefly outline calculation of EE for a single interval in a two dimensional CFT in presence of primary excitations. For details see [39–41].

We start with a CFT on a cylinder, parametrized by  $(x, \bar{x})$ , where  $x = \sigma + i\tau$  and  $\sigma \in [0, 2\pi]$ ,  $\tau \in (-\infty, +\infty)$ . The entangling surface,  $\mathcal{A}$ , is the interval  $x \in [a, b]$ . To prepare the system in a highest weight excited state, we generate the asymptotic *incoming* and *outgoing* states by inserting the corresponding operators  $\mathcal{O}(x, \bar{x})$  and  $\mathcal{O}^\dagger(x, \bar{x})$  at infinite past and future time, ( $\tau \rightarrow \mp\infty$ ), respectively. The quantity of interest is

$$tr \rho_{\mathcal{O}}^n \equiv \frac{Z_n}{Z_1^n} \lim_{x, \bar{x} \rightarrow -i\infty} \frac{\langle \prod_{k=0}^{n-1} \mathcal{O}_k(x, \bar{x}) \mathcal{O}_k^\dagger(-x, -\bar{x}) \rangle_{\mathcal{R}_n}}{\langle \mathcal{O}(x, \bar{x}) \mathcal{O}^\dagger(-x, -\bar{x}) \rangle_x^n}, \quad (8)$$

where the subscripts  $\mathcal{R}_n$  and  $x$  denote correlation on the Riemann surface with  $n$  cylinders and that on a single cylinder respectively. The subscript  $k$  labels the cylinder on which we have inserted the operators. We can now move over to the smooth  $z$  plane by

$$z = \left[ \frac{e^{-ix} - e^{-ia}}{e^{-ix} - e^{-ib}} \right]^{\frac{1}{n}}, \quad (9)$$

and use the transformation properties of the primary operators to calculate the relevant factors to arrive at

$$tr \rho_{\mathcal{O}}^n = \frac{Z_n}{Z_1^n} \mathcal{T} \frac{\langle \prod_{k=0}^{n-1} \mathcal{O}(z_k, \bar{z}_k) \mathcal{O}^\dagger(z'_k, \bar{z}'_k) \rangle_{\mathbb{C}}}{\langle \mathcal{O}(\zeta, \bar{\zeta}) \mathcal{O}^\dagger(\zeta', \bar{\zeta}') \rangle_{\mathbb{C}}^n}, \quad (10)$$

where the subscribe  $\mathbb{C}$  stands for the complex plane and  $\mathcal{T}$  is coming from the conformal transformation (9). Note that the points  $x = (a, b)$  map to  $z = (0, \infty)$ . In addition,  $z_k$  and  $z'_k$  are representing the insertion points,  $x = -i\infty$  and  $x = i\infty$  respectively, which lie on the circumference of a circle with unit radius on the complex plane  $\mathbb{C}$

$$\zeta = e^{i\pi\theta}, \quad \zeta' = e^{-i\pi\theta}, \quad z_k = e^{\frac{i\pi}{n}(\theta+2k)}, \quad z'_k = e^{\frac{i\pi}{n}(-\theta+2k)}, \quad k = 0, 1, \dots, n-1, \quad (11)$$

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<sup>3</sup>Look at transformations (9) and (15) as examples.

where  $\theta \equiv \frac{b-a}{2\pi} \equiv \frac{l}{2\pi}$ . The nontrivial information will be included in the transformation factor  $\mathcal{T}$  and the theory dependent correlation functions on the plane. The change in the EE coming from the insertions are calculable from the quantity

$$\mathcal{F}_O^{(n)} \equiv \frac{\text{tr} \rho_O^n}{\text{tr} \rho^n}. \quad (12)$$

### 3 Holographic Entanglement Entropy

In this section we will first state the *Ryu-Takayanagi's* proposal for a holographic description of EE. We then focus on two dimensional field theory living on a singular Riemann surface and review how its holographic space-time can be constructed. The latter will be used for a direct calculation of partition function.

#### 3.1 *Ryu-Takayanagi's* proposal

One interesting approach towards the calculation of the EE in a CFT, by applying the AdS/CFT correspondence is proposed in [9].

According to the *Ryu-Takayanagi's* proposal, one can find the EE for subregion  $\mathcal{A}$  in  $CFT_d$  holographically by using the following area law relation

$$S_{\mathcal{A}} = \min_{\gamma_{\mathcal{A}}} \left[ \frac{\text{Area}(\gamma_{\mathcal{A}})}{4G_N^{d+1}} \right], \quad (13)$$

where  $G_N$  is the Newton constant and  $\gamma_{\mathcal{A}}$  is a codimension two surface in the bulk geometry whose boundary coincides with the boundary of subregion  $\mathcal{A}$ ,  $\partial\gamma_{\mathcal{A}} = \partial\mathcal{A}$ .

As a simple example, where subregion  $\mathcal{A}$  is a single interval in a (1+1)-dim CFT, we must compute the length of the geodesic line in  $AdS_3$  space to find the EE. Then it is straightforward to show that this entropy exactly yields the known 2-dim CFT result.

An alternative approach for calculating the EE holographically which is more convenient for our purpose, is based on a direct holographic calculation to find the CFT partition function from the classical gravity one.

The following subsection provides a brief introduction to this method for a one dimensional single interval problem.

#### 3.2 Direct holographic calculation

In the remainder of this section, following [30], we review the direct calculation of Renyi entropy for the special case of a single interval in two dimensional CFT by holography.

The system lives on a plane, parametrized by  $(w, \bar{w})$  and the entangling space is the interval  $w \in [u, v]$ . This is related to the cylinder of section (2.2) by

$$w = e^{-ix}, \quad u = e^{-ia}, \quad v = e^{-ib}. \quad (14)$$

The starting point is the theory on  $\mathcal{R}_n$  that has been produced through replication and consists of  $n$  copies of the  $w$ -plane, glued together along the entangling space in a cyclic order. This produces a concentration of curvature at the boundaries of entangling subspaces. To give a holographic description of the problem we thus need to find a three dimensional space-time that has  $\mathcal{R}_n$  as the boundary. This can be achieved by the ‘‘holographic reconstruction of spacetime’’ [43], [44].

It is more convenient to work in the *Fefferman-Graham* (FG) gauge, mainly because all of the required nontrivial regularizations during the calculation of the EE take very simple forms in this gauge. The metric on the boundary is found through the map

$$z = \left( \frac{w - u}{w - v} \right)^{\frac{1}{n}}, \quad (15)$$

that takes  $\mathcal{R}_n$  to the flat  $z$ -plane. One finds that the metric is conformally flat,

$$ds^2 = \mathcal{C}(w, \bar{w}, u, v) \, dw d\bar{w}, \quad (16)$$

with the expected curvature at singular points found by  $R_0^w \sim \nabla_w^2 \log \mathcal{C}$ . Through trace anomaly, this in turn will be related to the expectation value of the boundary energy-momentum tensor,  $\langle T_{ww} \rangle \equiv -c/(24\pi) \mathcal{T}_{ww}$ , where  $c$  is the central charge of the CFT.

One can now use a suitable regularization to calculate the gravitational on-shell action which consists of the Einstein-Hilbert part and the boundary contributions. Putting everything together one arrives at

$$S_{grav} = -\frac{c}{12\pi} \int dw d\bar{w} \sqrt{g_{(0)}} \left( \frac{1}{4} R_0 \log \rho^* + \frac{1}{2} \sqrt{|\mathcal{T}_{ww}|^2} + \frac{3}{8} R_0 \right), \quad (17)$$

where  $\rho^*$  stands for the upper limit of the integral in the radial direction of space (see [30] for details). A careful calculation gives

$$S_{grav} = \frac{c}{6} \left( n - \frac{1}{n} \right) \log\left(\frac{l}{\delta}\right), \quad (18)$$

which, through eq.(4), results in the following expression for the Renyi entropy

$$S_n = \frac{1}{n-1} S_{grav} = \frac{c}{6} \left( 1 + \frac{1}{n} \right) \log\left(\frac{l}{\delta}\right). \quad (19)$$

In the last two expressions  $l = v - u$  and  $\delta$  is the short distance regulator around the singular points.

### 3.3 The role of the UV regulator

The results obtained above, as noticed in [30], may be attributed to a nontrivial regularization in the bulk integrations. The regulator surface extends the notion of the field theory short distance cut-off into the bulk by cutting-off the radial integration near the boundary. It also keeps the integrand away from the singularities at  $w = u$  and  $w = v$ . The (FG) gauge was used to simplify this procedure.

Alternatively, one can work in the usual Poincare coordinates

$$ds^2 = \frac{1}{r^2} (dr^2 + dz d\bar{z}), \quad (20)$$

and instead deform the usual *constant radius* regulator surface appropriately to obtain the former results. This can be seen by applying a suitable transformation between FG and Poincare coordinates, as follows [45]

$$r = \frac{\rho^{\frac{1}{2}} e^{-\psi}}{1 + \rho e^{-2\psi} |\partial_y \psi|^2}, \quad z = y + \partial_{\bar{y}} \psi \frac{\rho e^{-2\psi}}{1 + \rho e^{-2\psi} |\partial_y \psi|^2}, \quad (21)$$

where

$$y \equiv \left( \frac{w-u}{w-v} \right)^{\frac{1}{n}}, \quad e^\psi = \frac{n}{l} |w-u|^{(1-1/n)} |w-v|^{(1+1/n)} = \left| \frac{dw}{dy} \right|^2. \quad (22)$$

It is obvious that for  $\rho \sim \epsilon^2 \rightarrow 0$ , this is the desired conformal map between  $w$ -plane and its universal cover. The metric (20) in this new coordinate reads as

$$ds^2 = \frac{d\rho^2}{4\rho^2} + \frac{g_{ij}(\rho, w, \bar{w})}{\rho} dx^i dx^j, \quad (23)$$

where

$$i, j = 1, 2, \quad g_{ij}(\rho, w, \bar{w}) = g_{(0)ij} + \rho g_{(2)ij} + \dots, \quad g_{(0)ij} dx^i dx^j = dw d\bar{w}. \quad (24)$$

One can see that the auxiliary field  $\psi(z, \bar{z})$  is nothing but the *Liouville field*.

Now consider the following regulator surface

$$r = \epsilon e^{-\varphi(z, \bar{z})}, \quad e^{\varphi(z, \bar{z})} = nl \frac{|z|^{n-1}}{|z^n - 1|^2}. \quad (25)$$

Note that in the limit of  $\epsilon \ll 1$ ,  $\psi$  reduces to  $\varphi$  and hence  $e^{2\varphi} = |dw/dz|^2$ . It can be shown that the induced metric on this surface has the same singular conformal factor  $\mathcal{C}$  in (16). Using this regulator, we find that the gravitational on-shell action in the Poincare coordinate will only depend on the auxiliary scalar field  $\varphi(z, \bar{z})$  through (see [30] and [45])

$$S_{grav} = -\frac{c}{48\pi} \int dz d\bar{z} (\partial\varphi \bar{\partial}\varphi - 4\partial\bar{\partial}\varphi). \quad (26)$$

Substituting the explicit form of  $\varphi(z, \bar{z})$  in this expression one can reproduce the former results in the (FG) gauge, eq.(18). We will use this approach in our holographic calculations of the following section.

## 4 Holographic calculation of the entanglement entropy for excited states

In this section we present our main calculations and results of this paper. As explained before, we are interested in a general CFT on a circle which has been excited by primary operators. We would like to calculate the EE of a single interval in presence of these excitations. In the following we present a holographic description of this problem which has already been addressed in a field theoretic setup and which has been shortly reviewed in section (2.2).

### 4.1 Holographic Setup

Start with the CFT on the infinite  $x$ -cylinder of section (2.2), with insertions at infinite past and future time to account for excitations. The replicated theory consists of  $n$  such cylinders, glued cyclically along the entangling interval, denoted as before by  $\mathcal{R}_n$ . This also produces  $n$  copies of operator insertions for in and out states each.

The objective is to calculate the path integral on  $\mathcal{R}_n$  by holography. This will give us the Renyi, and subsequently, the entanglement entropy. One can try to find a gravitational background which has this replicated cylinder at the boundary. The operator insertions will amount to turning on appropriate fields in the background. The gravitational partition function thus found will be, according to AdS/CFT, our desired result. This program is in principle possible through the



“holographic reconstruction of space-time” and would be a generalization of section (3.2) to the case with excitations.

We will choose, however, to take a different route and find the gravitational analogue of equation (7). That is, we prefer to deal with the CFT on the smooth manifold,  $\mathcal{M}_C$ , and worry about the complexities of the replicated cylinder only through the transformation  $\mathcal{R}_n \rightarrow \mathcal{M}_C$ . This choice will give us the advantage that we should now look for, and work with, a gravitational background that will have a smooth boundary rather than a singular one. The nontrivial information of  $\mathcal{R}_n$ , and accordingly those in the transformation to  $\mathcal{M}_C$ , will be encoded in the regularization scheme that we choose to calculate the gravitational partition function along the lines of section (3.3).

The setup will thus be as follows; the flat complex  $z$ -plane, also denoted by  $\mathbb{C}$ , is our  $\mathcal{M}_C$  and sits on the boundary with insertion points given by (11). The transformation  $\mathcal{R}_n \rightarrow \mathcal{M}_C$  is nothing but (9). The gravitational background will thus be pure  $AdS_3$  on which we will turn on a scalar field with suitable scaling properties and boundary conditions to account for the excitations. We will then use the map (9) to find the *Liouville* field that determines the shape of our regulator surface. We use this regulator to find the renormalized on-shell gravitational and matter actions. The standard recipe of AdS/CFT will then give us the exact one point function of the boundary operator and the desired correlation functions.

As a matter of simplifications, we map the replicated  $x$ -cylinders to replicated  $w$ -planes of section (3.2) by the map of (14). In the case of possible confusion, we differentiate between the two by  $x$  and  $w$  superscripts respectively. The link between the replicated  $w$ -planes with the  $z$ -plane will be provided by (15). As a final step we will map the  $z$ -plane to cylinder again. Schematically

$$\mathcal{R}_n^{(x)} \xrightarrow{w=e^{-ix}} \mathcal{R}_n^{(w)} \xrightarrow{eq. (15)} z\text{-plane} \xrightarrow{s=-i \ln z} s\text{-cylinder} , \quad (27)$$

such that in the end we have a map between a replicated cylinder to a smooth one. We have introduced the intermediate manifolds, i.e.,  $\mathcal{R}_n^{(w)}$  and  $z$ -plane (or  $\mathbb{C}$ ) to simplify the holographic calculations and it is in fact this step  $\mathcal{R}_n^{(w)} \rightarrow \mathbb{C}$  that we will take by holography in the following section.

The quantities we will find with the nontrivial regularization will be in terms of those obtained with the standard simple cutoff regulator. The former quantities are identified with CFT correlators on  $\mathcal{R}_n$  and the latter with those on  $\mathcal{M}_C$  which is simply the complex plane  $\mathbb{C}$ . This will in fact be the holographic analogue of going from equation (8) to (10) in section (2.2). The role of the linking equation (9) in field theory will be played by the *Liouville* field and the regulator surface in gravity. In the next section we will show how this works.

## 4.2 Calculations

We use the standard method of “holographic renormalization” [44] (see also [46] for review and references) to extract physical quantities of the boundary CFT out of the bulk. The quantity we should compute consists of two parts

$$S = S_g + S_m ,$$

where  $S_g$  and  $S_m$  refer to the gravitational and matter actions respectively. We will use two sets of coordinates and metrics for the space which are collectively denoted by  $(x^\mu, G_{\mu\nu})$ . One is the usual Poincare coordinates  $(r, z, \bar{z})$  with the metric (20) and the other one is the FG coordinates  $(\rho, w, \bar{w})$  with the metric (23). These two are related by (21).

### Regulators

Physical quantities are obtained upon calculating the renormalized actions in bulk. This in turn requires a regularization scheme. A surface at a constant Poincare radius,  $r = \epsilon$ , which we will denote by  $\mathcal{M}_0$  will be used to calculate the physical quantities of the CFT on the  $z$ -plane,  $\mathbb{C}$ . The regulator surface of (25), denoted by  $\mathcal{M}$ , is attributed to the CFT on  $\mathcal{R}_n$ . A closely related surface which will play a crucial role for us is obtained as a constant radius surface in the FG coordinates,  $\rho = \epsilon^2$ , and which we will denote by  $\mathcal{M}'$ . To summarise our notation

$$r = \epsilon \equiv \mathcal{M}_0, \quad r = \epsilon e^{-\varphi} \equiv \mathcal{M} \quad \rho = \epsilon^2 \equiv \mathcal{M}' . \quad (28)$$

The objective is to calculate the renormalized action on  $\mathcal{M}$  and this can be achieved by going through the complete process of holographic renormalisation. This is in principle doable but recall that in the limit of  $\epsilon \ll 1$  the regulators  $\mathcal{M}$  and  $\mathcal{M}'$  are equivalent and will therefore result in identical physical quantities. The fact that  $\mathcal{M}'$  has a simple form in the FG coordinates facilitates the regularization process. We will take advantage of this fact and use the well studied renormalized action on  $\mathcal{M}'$ .

The surface  $\mathcal{M}$ , on the other hand, has the advantage that as compared to  $\mathcal{M}'$  has a simpler description in the Poincare coordinates. This will enable us to write down the fields on this surface in terms of those in the Poincare coordinates in a simple way which will eventually lead to a clear relation between physical quantities on  $\mathcal{M}$  and those on  $\mathcal{M}_0$ . This, in turn, will be used to describe the field theory on  $\mathcal{R}_n$  in terms of the theory on  $\mathbb{C}$ . We will expand on the details in the course of the calculations.

The plan therefore will be as the following. We will write the renormalized action on  $\mathcal{M}_0$  and extract from it the quantities attributed to the theory on  $\mathbb{C}$ . We will then find the renormalized action on  $\mathcal{M}'$  and use perturbative methods to approximate the renormalized action on  $\mathcal{M}$ . The latter will be used to extract physical objects of the theory on  $\mathcal{R}_n$  which are related to those on  $\mathbb{C}$  in a simple way.

### Action

The gravitational action  $S_g$  using the nontrivial regulator  $\mathcal{M}$  has already been studied in [30] and their results hold here. We will thus focus on the scalar part. The action for a free scalar in bulk is given by

$$S_m = \frac{1}{2} \int d^3x \sqrt{G} (G^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + m^2 \Phi^2) . \quad (29)$$

The scalar field in the Poincare and FG coordinates are denoted by  $\Phi$  and  $\Phi'$  and are related by

$$\Phi(r, z, \bar{z}) = \Phi'(\rho, w, \bar{w}) . \quad (30)$$

The asymptotic behaviour of the scalar in the Poincare coordinate is given by

$$\Phi(r, z, \bar{z}) = r^{2-\Delta} \phi(r, z, \bar{z}) , \quad \phi(r, z, \bar{z}) = \phi_0(z, \bar{z}) + r^2 \phi_2(z, \bar{z}) + \dots \quad (31)$$

where, dots stand for terms with higher powers as well as logarithmic terms in  $r$ . Equivalently we can expand the scalar in the FG coordinates as

$$\Phi'(\rho, w, \bar{w}) = \rho^{\frac{2-\Delta}{2}} \phi'(\rho, w, \bar{w}) , \quad \phi'(\rho, w, \bar{w}) = \phi'_0(w, \bar{w}) + \rho \phi'_2(w, \bar{w}) + \dots \quad (32)$$

Substituting any of the two expressions in the scalar field equation results in the familiar relation between the mass of the scalar field and the conformal dimension of the dual operator,  $\Delta$ , as  $m^2 = \Delta(2 - \Delta)$ .

Let us first take the regulator surface to be  $\mathcal{M}_0$ . The *regularized* matter action is obtained when the integration region is bounded by this surface

$$S_{m,reg} = \frac{1}{2} \int_{r \geq \epsilon} d^3x \sqrt{G} (G^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + m^2 \Phi^2) = -\frac{1}{2} \int_{r=\epsilon} \sqrt{G} dz d\bar{z} G^{rr} \Phi \partial_r \Phi = -\frac{1}{2} \int_{r=\epsilon} \sqrt{\gamma} dz d\bar{z} r \Phi \partial_r \Phi, \quad (33)$$

where  $\gamma_{ij} = 1/r^2 \delta_{ij}$  is the induced metric on a constant  $r$  surface. Also note that we have used the equations of motion to drop the bulk action. The *subtracted* action is defined upon the addition of suitable counterterms to the regularized action<sup>4</sup>

$$S_{sub}(\Phi, \gamma)_{\mathcal{M}_0} = \int_{r=\epsilon} \sqrt{\gamma} dz d\bar{z} \Phi \left[ -\frac{r}{2} \partial_r \Phi + \frac{2-\Delta}{2} \Phi + \frac{1}{2(\Delta-2)} \square_\gamma \Phi \right], \quad (34)$$

where  $\square_\gamma$  stands for the Laplacian operator made by the induced metric and we have dropped some higher derivative terms in the counterterm part. Similarly on the regulator surface  $\mathcal{M}'$  we find<sup>5</sup>

$$S_{sub}(\Phi', \gamma')_{\mathcal{M}'} = \int_{\rho=\epsilon^2} \sqrt{\gamma'} dw d\bar{w} \Phi' \left[ -\rho \partial_\rho \Phi' + \frac{2-\Delta}{2} \Phi' + \frac{1}{2(\Delta-2)} \square_{\gamma'} \Phi' \right], \quad (35)$$

where  $\gamma'_{ij} = g_{ij}(\rho, w, \bar{w})/\rho$  and we are using “primed” quantities to emphasise that on a constant  $\rho$  surface everything is naturally written in terms of the FG coordinates and  $\Phi'$ .

### $\mathcal{M}'$ vs. $\mathcal{M}$

We now wish to use (35) to calculate the renormalized action on  $\mathcal{M}$ . In order to do this, we should expand (35) in the limit of  $\epsilon \ll 1$ . Note that in this limit the transformation (21) reduces to

$$r = \rho^{1/2} e^{-\varphi}, \quad z = \left( \frac{w-u}{w-v} \right)^{1/n}. \quad (36)$$

The relation between  $z$  and  $w$  now has no  $\rho$  dependence and describes the conformal transformation from  $\mathcal{R}_n$  to  $\mathbb{C}$ . Also  $\rho = \epsilon^2$ , or equivalently  $r = \epsilon e^{-\varphi}$ , is now describing the regulator surface of  $\mathcal{M}$ . On this surface we can approximate the scalar field  $\Phi'$  by  $\tilde{\Phi}(\rho, w, \bar{w}) = \Phi(r, z, \bar{z})$ . We are using a  $\tilde{\Phi}$  instead of  $\Phi'$  and call it an approximation because we are using the approximate transformation (36). Since the  $\epsilon$  dependence is now entering only through the relation between  $r$  and  $\rho$ , the coefficients of an  $\epsilon$  expansion can be easily extracted and compared in the two coordinates

$$\tilde{\Phi}(\epsilon^2, w, \bar{w}) = \Phi(\epsilon e^{-\varphi}, z, \bar{z}) \Rightarrow \tilde{\phi}_{2n} = (e^{-\varphi})^{2n+2-\Delta} \phi_{2n}, \quad (37)$$

where

$$\tilde{\Phi}(\rho, w, \bar{w}) = \rho^{\frac{2-\Delta}{2}} \tilde{\phi}(\rho, w, \bar{w}), \quad \tilde{\phi}(\rho, w, \bar{w}) = \tilde{\phi}_0(w, \bar{w}) + \rho \tilde{\phi}_2(w, \bar{w}) + \dots \quad (38)$$

We can now write down the approximate form of (35) by replacing  $\Phi'$  with  $\tilde{\Phi}$  and remembering that the transformation to the Poincare coordinates is now given by (36). We denote the resulting approximate action by  $\tilde{S}_{sub}(\tilde{\Phi}, \gamma')_{\mathcal{M}}$ . Again the tilde is there to remind us of the approximation.

The action  $\tilde{S}$  thus found is a good approximation to the subtracted action on the regulator surface  $\mathcal{M}$  for small  $\epsilon$  and for a sufficiently slowly varying  $\varphi$ . In principle there will be corrections

<sup>4</sup>For the details of this procedure see [44].

<sup>5</sup>We avoid singular points on the surface by limiting the integration to the regular parts. This will have no effect for the matter action and we thus do not mention it in the integration limits. As for the gravitational action the singular points play a crucial role (see [30]).

in derivatives of  $\varphi$  such that physical quantities constructed from  $\tilde{S}$  yield finite values. We ignore these corrections in the following and will comment on the validity of our assumption later.

Note that the action on  $\mathcal{M}'$  is an exact one but a simple relation like (37) is missing in that case. Conversely, such a relation exists for  $\mathcal{M}$  but instead the action we found is approximate.

### Interpretation of $\tilde{S}$

The action  $\tilde{S}_{\mathcal{M}}$  can be interpreted as the familiar holographic description of a conformal transformation on the boundary (deformed) CFT. Instead of the usual constant scaling in the radial AdS direction, we have been using a local scaling factor which has been forced on us by the geometry of  $\mathcal{R}_n$ . Let us see how this works.

Recall that conformal transformation is a sequence of a Weyl scaling of the metric followed by a coordinate transformation. Once we rewrite  $\tilde{S}_{\mathcal{M}}$  in terms of quantities and fields which are naturally defined in the Poincare coordinates we will have described the Weyl scaling. For this purpose we may first easily check that for small  $\epsilon$  and on  $\mathcal{M}$

$$\sqrt{\gamma'} dwd\bar{w} = \sqrt{\tilde{\gamma}} dzd\bar{z}, \quad \rho \partial_\rho \tilde{\Phi} = \frac{r}{2} \partial_r \Phi, \quad \square_{\gamma'} \tilde{\Phi} = \square_{\tilde{\gamma}} \Phi, \quad (39)$$

where  $\tilde{\gamma}_{ij} = \gamma_{ij} e^{2\varphi}$ . Plugging these in  $\tilde{S}$  we find that

$$\tilde{S}_{sub}(\tilde{\Phi}, \gamma')_{\mathcal{M}} = \int_{\mathcal{M}} \sqrt{\tilde{\gamma}} dzd\bar{z} \Phi \left[ -\frac{r}{2} \partial_r \Phi + \frac{2-\Delta}{2} \Phi + \frac{1}{2(\Delta-2)} \square_{\tilde{\gamma}} \Phi \right] = S_{sub}(\Phi, \tilde{\gamma})_{\mathcal{M}}. \quad (40)$$

It can be seen that  $S_{sub}(\Phi, \tilde{\gamma})_{\mathcal{M}}$ , thus obtained, is nothing but  $S_{sub}(\Phi, \tilde{\gamma})_{\mathcal{M}_0}$  with a scaled metric and of course with a different argument for  $\Phi$ . To complete the conformal transformation we now make a coordinate transformation that cancels out the Weyl factor of the metric. This is of course the transformation  $(z, \bar{z}) \rightarrow (w, \bar{w})$  in (36) which takes us back to  $S_{sub}(\Phi, \tilde{\gamma})_{\mathcal{M}}$ .

### One point function

We can now extract the exact one point function of the field theory operator that is dual to the bulk scalar field. As for the theory on  $\mathbb{C}$ , this is obtained as <sup>6</sup>

$$\langle \mathcal{O}(z, \bar{z}) \rangle_{\mathbb{C}} = \lim_{\epsilon \rightarrow 0} \left[ \frac{1}{r^\Delta \sqrt{\gamma}} \frac{\delta S_{sub}(\Phi, \gamma)}{\delta \Phi} \right]_{\mathcal{M}_0} = (2-2\Delta) \phi_{(2\Delta-2)}. \quad (41)$$

The one point function for the CFT on  $\mathcal{R}_n$  can be obtained either from  $S_{sub}(\Phi', \gamma')_{\mathcal{M}'}$  in (35) or, equivalently, from the subtracted action on  $\mathcal{M}$ . Using the former we get

$$\langle \mathcal{O}'(w, \bar{w}) \rangle_{\mathcal{R}_n} = \lim_{\epsilon \rightarrow 0} \left[ \frac{1}{\rho^{\Delta/2} \sqrt{\gamma'}} \frac{\delta S_{sub}(\Phi', \gamma')}{\delta \Phi'} \right]_{\mathcal{M}'} = (2-2\Delta) \phi'_{(2\Delta-2)}. \quad (42)$$

This is an exact result. To write it in terms of  $\langle \mathcal{O}(z, \bar{z}) \rangle_{\mathbb{C}}$  we need to write  $\phi'_{(2\Delta-2)}$  in terms of  $\phi_m$ 's which can be done perturbatively. Instead, we use the approximate action  $\tilde{S}_{sub}(\tilde{\Phi}, \gamma')_{\mathcal{M}}$  which yields

$$\langle \mathcal{O}'(w, \bar{w}) \rangle_{\mathcal{R}_n} = \lim_{\epsilon \rightarrow 0} \left[ \frac{1}{\rho^{\Delta/2} \sqrt{\gamma'}} \frac{\delta \tilde{S}_{sub}(\tilde{\Phi}, \gamma')}{\delta \tilde{\Phi}} \right]_{\mathcal{M}} = (2-2\Delta) \tilde{\phi}_{(2\Delta-2)}. \quad (43)$$

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<sup>6</sup>We should note that the variations of the regularized parts of the subtracted actions are taken from the regularized bulk actions before imposing the equations of motion, i.e., from the first line in (33).

The relation to  $\langle \mathcal{O}(z, \bar{z}) \rangle_{\mathbb{C}}$  is now obvious after consulting (37)

$$\langle \mathcal{O}'(w, \bar{w}) \rangle_{\mathcal{R}_n} = e^{-\varphi \Delta} \langle \mathcal{O}(z, \bar{z}) \rangle_{\mathbb{C}} . \quad (44)$$

Let us summarise what we have done. CFT on  $\mathcal{R}_n$  is claimed to be obtained from a regularization of  $AdS$  space on  $\mathcal{M}$ . Since in the asymptotic region,  $\mathcal{M}$  and  $\mathcal{M}'$  become equivalent we could use the latter as well.

Since  $\mathcal{M}$  is related to  $\mathcal{M}_0$  by an exact conformal transformation, fields on the former are obviously related to those on the latter (see (37)). This makes it easy to relate physical quantities that are extracted from them. Instead, we have to work out the details of holographic renormalisation method with the regulator  $\mathcal{M}$  which is in principle a difficult task.

On the other hand, since  $\mathcal{M}'$  has a simple form in the FG coordinates, we already know the form of renormalized action when using it as a regulator. However, the transformation from  $\mathcal{M}'$  to  $\mathcal{M}_0$  is a complicated one and reduces to our desired conformal transformation only in the limit. This in turn makes it difficult to relate physical quantities extracted from  $\mathcal{M}'$  to those from  $\mathcal{M}_0$ .

In the above, we have used the form of renormalized action on  $\mathcal{M}'$  to find that on  $\mathcal{M}$  approximately. The final result (44) has a suggestive form which guides us to the interpretation and/or justification of the approximations we have made.

#### Interpretation of the approximations

We would now like to see how good an approximation we have used to have replaced  $S_{\mathcal{M}'}$  with  $\tilde{S}_{\mathcal{M}}$ . A precise answer to this question requires a complete analysis of holographic renormalisation with the regulator  $\mathcal{M}$  or, equivalently, working out a series expansion in  $\epsilon$  which relates  $\phi'_m$  and  $\phi_n$  based on (30) and using the exact transformation (21). For now, we will make use of our answer (44) which was obtained approximately.

Remember that upon a conformal transformation  $z \rightarrow w$ , there are certain operators, primaries, which behave in a simple form

$$\mathcal{O}(z, \bar{z}) \rightarrow \mathcal{O}'(w, \bar{w}) = \left(\frac{\partial z}{\partial w}\right)^h \left(\frac{\partial \bar{z}}{\partial \bar{w}}\right)^{\bar{h}} \mathcal{O}(z, \bar{z}) , \quad (45)$$

where  $h$  and  $\bar{h}$  are conformal weights of the primary operator. Comparing this with (44) and recalling that  $\Delta = h + \bar{h}$ , we have found the conformal properties for a scalar primary operator by holography. We may then conclude that our approximations are equivalent to assuming the bulk scalar to be dual to a field theory primary. One can reverse the argument and conclude that if we require our final result to be exact and unaffected by corrections we should then impose constraints on the asymptotic expansion of the bulk field and hence further relations between the  $\phi_n$ 's. These restrictions can be interpreted as the bulk definition for a CFT primary operator.

Preliminary calculations show that our final result should be corrected by  $\phi_{2n}$  terms with  $2n < 2\Delta - 2$  in a generic case. Recalling that any operator can be expanded in the basis of primaries, we may restate our approximation as an expansion of conformal properties of a generic operator in terms of those of the basis operators. The result (44) may then be considered as the term coming from the primary operator in the expansion basis that has the highest dimension. This problem obviously needs further investigation which is in progress.

#### n-point function

Recall that in the asymptotic expansion of the scalar field,  $\phi_0$  is interpreted as the external source for the boundary field theory. We can thus restate our result, (44), as following: in order to

find the exact one point function on  $\mathcal{R}_n$ , one can use the theory on  $\mathbb{C}$  but with a rescaled external source

$$\phi_0 \rightarrow e^{\varphi\Delta} \phi_0 . \quad (46)$$

This relation may seem confusing as  $\phi'_0 = (e^\varphi)^{\Delta-2}$  but one should note that the factor of  $e^{-2\varphi}$  is cancelled out by its inverse from the volume element  $|dw/dz|^2 = e^{2\varphi}$ . We expect this result to remain valid when we turn on interaction terms for the scalar in the bulk and couple the system to gravity. The one point function will definitely be affected by interactions but what remains unchanged is how a conformal transformation from  $\mathcal{R}_n$  to  $\mathbb{C}$  is applied on it.

Recall that in order to find the  $n$ -point function we should take  $n-1$  derivatives from the exact one point function with respect to the external source. The rescaling of (46) will thus lead to

$$\prod_{i=1}^n \frac{\delta}{\delta\phi_0(z_i, \bar{z}_i)} \rightarrow \prod_{i=1}^n e^{-\varphi(z_i, \bar{z}_i)\Delta} \frac{\delta}{\delta\phi_0(z_i, \bar{z}_i)} . \quad (47)$$

### EE for excitations

We can finally use our gravitational results to write (8), which is defined on a singular cylinder, in the form of correlators on a smooth cylinder. For the intermediate manifolds,  $\mathcal{R}_n^{(w)}$  and  $\mathbb{C}$ , and according to the recipe (47), we can write

$$\lim_{\substack{w \rightarrow 0 \\ w' \rightarrow \infty}} \frac{\langle \prod_{k=0}^{n-1} \mathcal{O}_k(w, \bar{w}) \mathcal{O}_k^\dagger(w', \bar{w}') \rangle_{\mathcal{R}_n}}{\langle \mathcal{O}(w, \bar{w}) \mathcal{O}^\dagger(w, \bar{w}) \rangle_w^n} = \frac{\prod_{k=0}^{n-1} e^{-[\varphi(z_k, \bar{z}_k) + \varphi(z'_k, \bar{z}'_k)]\Delta}}{(|(\zeta-1)(\zeta'-1)|/l)^{2n\Delta}} \frac{\langle \prod_{k=0}^{n-1} \mathcal{O}(z_k, \bar{z}_k) \mathcal{O}^\dagger(z'_k, \bar{z}'_k) \rangle_{\mathbb{C}}}{\langle \mathcal{O}(\zeta, \bar{\zeta}) \mathcal{O}^\dagger(\zeta', \bar{\zeta}') \rangle_{\mathbb{C}}^n} , \quad (48)$$

where  $w = 0$  and  $w' = \infty$  are the location of in and out insertions on the  $w$ -plane respectively. Transformations factors for  $\mathcal{R}_n^{(w)} \rightarrow \mathcal{R}_n^{(x)}$  and  $\mathbb{C} \rightarrow s$ -cylinder, are both equal to one and once we plug the values of arguments  $z_k, \dots$  in the above expression, we arrive at our final result

$$\mathcal{F}_{\mathcal{O}}^{(n)} \equiv \frac{tr \rho_{\mathcal{O}}^n}{tr \rho^n} = n^{-2n(h+\bar{h})} \frac{\langle \prod_{k=0}^{n-1} \mathcal{O}(s_k, \bar{s}_k) \mathcal{O}^\dagger(s'_k, \bar{s}'_k) \rangle_{cy}}{\langle \mathcal{O}(\xi, \bar{\xi}) \mathcal{O}^\dagger(\xi', \bar{\xi}') \rangle_{cy}^n} , \quad (49)$$

where

$$\xi = \pi\theta , \quad \xi' = -\pi\theta , \quad s_k = \frac{\pi}{n}(\theta + 2k) , \quad s'_k = \frac{\pi}{n}(-\theta + 2k) , \quad k = 0, 1, \dots, n-1 . \quad (50)$$

This is in complete agreement with the field theory result.  $EE$  is now immediately obtained by calculating

$$S = -\partial/\partial_n tr \rho_{\mathcal{O}}^n|_{n=1} . \quad (51)$$

Noting that  $tr \rho_{\mathcal{O}} = \mathcal{F}_{\mathcal{O}}^{(1)} = 1$ , we find

$$S_{\mathcal{O}} = S_{GS} - \frac{\partial}{\partial_n} \mathcal{F}_{\mathcal{O}}^{(n)}|_{n=1} , \quad (52)$$

where the subscript GS stands for the ground state of the theory.

## 5 Discussion

In this work we have studied the entanglement entropy for excited states in a two dimensional conformal field theory on a circle. The entangling area is a single interval and the excited states are produced by vertex operators. We have used holography to address this problem and we were able to reproduce the exact field theory results for primary excitations.

In a field theory setup, this problem boils down to the calculation of certain correlation functions on a singular manifold. Since in two dimensions all metrics are conformally equivalent, one can encode any singularity in the conformal structure of the space. Consequently, the problem reduces to a standard calculation on a smooth manifold. We simulate the same ideology in the holographic language.

Two dimensional CFT's are conjectured to be described by asymptotically  $AdS_3$  spaces. Such spaces are always locally  $AdS$  no matter of whatever modifications we make. We thus expect that even if the field theory is living on a singular manifold, all the complexities can be encoded in an appropriately chosen regulator surface in the bulk. Regulator surfaces are all conformally equivalent and we choose the conformal class that has the previously mentioned singular structure.

As a point of reference, we also use the usual smooth regulator surface which is defined at a constant radius of the Poincare coordinates. We show that switching between the two regulators amounts to performing a conformal transformation on the dual field theory. This enables us to make a relation between physical quantities that we compute using the two regulation schemes. We use perturbative methods to draw such a comparison and conclude that our approximations are equivalent to assuming certain conformal properties for the field theory operators.

This approach should in principle enable us to find a bulk definition for conformal properties of the field theory objects, and in particular, the definition for a field theory primary operator. This subject is currently under study and we postpone it to a future work.

There are many natural extensions to the present work. An immediate one is the case with a finite temperature. A generalisation to higher dimensions is also of particular interest. Another interesting direction is the recently discovered thermodynamic properties for entanglement entropy of excited states in certain limits [47]. The holographic setup can in principle be used to explore such properties in more generality.

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